Chapter 3 – Probability

An *experiment* is a procedure by which an observation or measurement is obtained. An execution is called a *trial*. The observation or measurement obtained is called the *outcome* of the trial. A *random experiment* is one where the outcomes depend upon chance. The set of all possible outcomes of an experiment is called the *outcome set* or *sample space*, and denoted *S*.

A. Examples of *dichotomous experiments* include

(1) flipping a two-sided coin, S = {H,T};

(2) drawing one card from a deck of cards (13 ♠'s, 13 ♣'s, 13 ♥'s, 13 ♦'s,) and looking for a ♦, S = { ♦, not ♦};

(3) rolling a die and looking for a '5' or '6', S = {'5' or '6', not};

(4) choosing a uniform random number from the interval I = [0,1] and noting whether or not it is in the sub-interval I_S = [½,¾], S = {in I_S, not in I_S}

B. Example of a *discrete experiment* – draw n = 3 cards from a deck of cards (with replacement) and count the number of diamonds.

C. Example of a *continuous experiment* – choose a uniform random number in the interval [0,1] and report the number.

Note the interesting difference between the random variables discussed in Example 3.1 (p.81) – Geiger counts so S = {0, 1, 2 … } – and Example 3.2 (p.82) – the time between Geiger clicks so S = (0 , ∞).

If *event* E = a '5' or '6' in A-3 above, then the probability of E is

$$\boxed{\Pr\{E\} = 2/6 = 1/3,}$$

since there are 2 *favorable outcomes* out of 6 *possible outcomes*.

Similarly, for example B above (cf. Ex. 3.17 on pp.85-6), π = ¼ and

Pr(0) = Pr(FFF)	$= (1-\pi)^3$	= 0.421875
Pr(1) = Pr(SFF,FSF,FFS)	$= 3\pi(1-\pi)^2$	= 0.421875
Pr(2) = Pr(SSF,SFS,FSS)	$= 3\pi^2(1-\pi)$	= 0.140625
Pr(3) = Pr(SSS)	$= \pi^3$	= 0.015625

These results are based on *Theoretical Probability* (p.83), to which we can add *Empirical Probability* (p.86), Simulation or *Monte Carlo* results (p.88), and *Subjective Probability* (p.89).

<u>Basic Probability Rules</u> include – For events E and F,

1. **$0 \leq \Pr(E) \leq 1$, $\Pr(\emptyset) = 0$, $\Pr(S) = 1$, and $\Pr(\text{not } E) = 1 - \Pr(E)$.**
2. **If $E \subset F$, then $\Pr(E) \leq \Pr(F)$.**
3. ***<u>The Addition Rule</u>*: $\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(E \cap F)$ with extensions (p.96); $\cup$ = *union* = 'or' and $\cap$ = *intersection* = 'and'.**
4. ***<u>The Multiplication Rule</u>*: $\Pr(E \cap F) = \Pr(E) \times \Pr(F \mid E) = \Pr(F) \times \Pr(E \mid F)$ where $\Pr(F \mid E)$ is the *conditional probability* of event F given E has occurred.**
5. **E and F are *independent* if $\Pr(E \mid F) = \Pr(E)$ or equivalently if $\Pr(F \mid E) = \Pr(F)$; for independent events, the multiplication rule is therefore $\Pr(E \cap F) = \Pr(E) \times \Pr(F)$.**
6. **E and F are *disjoint* if $\Pr(E \cap F) = 0$.**

<u>Examples</u>: In a deck of playing cards, draw one card, and let E = {get a diamond} = {2♦, 3♦, 4♦, 5♦, 6♦, 7♦, 8♦, 9♦, 10♦, J♦, Q♦, K♦, A♦}, F = {get a picture card} = {J♠,J♣,J♥,J♦,Q♠,Q♣,Q♥,Q♦,K♠,K♣,K♥,K♦}, and G = {get a black card}. Then,

- **Pr(E) = ¼ and Pr(F) = 12/52 = 3/13, each between 0 and 1**
- **Pr{'not G'} = Pr{red card} = ½**
- **E ⊂ 'not G' illustrates #2 above, and indeed ¼ ≤ ½**
- **Pr(E ∩ F) = Pr(J♦, Q♦, K♦) = 3/52 (so E and F are not disjoint)**
- **By #3 above, Pr(E ∪ F) = Pr(♦ or picture card)**
  = 13/52 + 12/52 - 3/52 = 22/52 = 11/26
- **E and F are independent here since Pr(F | E) = Pr(F) since 'F | E' is getting a picture card given the card is a diamond**
- **E and G are disjoint (since diamonds are not black) and cannot be independent since Pr(E ∩ G) = 0. Also, Pr(E ∪ G) = Pr(E) + Pr(G) = ¼ + ½ = ¾**

Tree Diagrams **are very helpful to calculate compound probabilities. Consider drawing 2 cards *without replacement* from a deck of cards and recording whether or not the card is a diamond. (D_1 = draw 1.)**

$D_1 = Y$ ($p = 13/52$)

- **$D_2 = Y$ ($p = 12/51$)   Probability = (13/52)*(12/51)**
- **$D_2 = N$ ($p = 39/51$)   Probability = (13/52)*(39/51)**

$D_1 = N$ ($p = 39/52$)

- **$D_2 = Y$ ($p = 13/51$)   Probability = (39/52)*(13/51)**
- **$D_2 = N$ ($p = 38/51$)   Probability = (39/52)*(38/51)**

Note that these probabilities differ from the respective (Binomial) probabilities we would get if this selection was done *with replacement*: (13/52)*(13/52), (13/52)*(39/52), (39/52)*(13/52), and (39/52)*(39/52).

Number of ♦ in n = 2 draws	Probability without replacement	Probability with replacement
0	0.5588	0.5625
1	2*0.1912 = 0.3824	2*0.1875 = 0.3750
2	0.0588	0.0625

Bayes' Method **(pp.107-111) is an indispensable tool to help us reverse the order in conditional probabilities. In the above Tree, suppose that in the first branch of the Tree, we list the *Prevalence* (top branch) or otherwise (lower branch) of a disease. In the second branch, we give the respective probabilities (for those with and without the disease) of a Test (for the disease) giving a positive result. The *Sensitivity* of the Test is the probability of a positive test result given**

that a chosen individual has the disease; the *Specificity* of the Test is the probability of a negative test result given that a chosen individual does not have the disease. Bayes Rule is used to find PV^+ = Pr{disease|positive}, the probability that an individual who tests positive actually has the disease. Indeed,

PV^+ = Pr{disease|positive} = Pr{ disease and positive } / Pr{positive}

Let π = Pr{disease}, π_1 = the Sensitivity, and π_2 = 1 - the Specificity, then

$$PV^+ = \frac{\pi \times \pi_1}{(\pi \times \pi_1) + [(1-\pi) \times \pi_2]} = \frac{1}{1 + \theta \frac{1-\pi}{\pi}}, \text{ where } \theta = \pi_2 / \pi_1.$$

An extension to multiple branches is given on pp. 110-111.

Example (Mammography) from p.109 – Prevalence of BC (breast cancer) in this population is assumed to be known and reported to be π = 0.0004. The Sensitivity (π_1) and Specificity (1 - π_2) for a particular BC test are reported to be 0.80 and 0.90 respectively. PV^+, the probability that a woman who tests positive actually has BC, is then calculated to be

$$PV^+ = \frac{(0.0004) \times (0.80)}{(0.0004) \times (0.80) + (0.9996) \times (0.10)} = 0.003191065.$$

Note here how different are the following:

(1) the probability that a woman who tests positive actually having BC (0.32%),

(2) the probability that a randomly selected woman having BC (0.04%), and

(3) the probability that a woman who has BC testing positive (80%).