# Chapter 9 – Waiting Time Random Variables

**Here, we look at two discrete distributions (Geometric and Poisson) and one continuous distribution (Exponential) in greater depth.**

**Geometric RV's – In Chapter 4 (p.126 ff) we looked at one variant – here, we look at another one. Here, T = *number of trials until the first success*; previously, X = number of *failures* until the first success. The pmf for T is**

$$f_T(t) = \pi(1-\pi)^{t-1},\ t = 1, 2, 3 \ldots,$$

**where π is the 'success' probability (for a single trial); $f_T(t) = 0$ for all other values of t. It follows that the cdf of T is $F(t) = \Pr\{T \le t\} = 1 - (1-\pi)^t$, and the *survival function* of T is $S(t) = \Pr\{T > t\} = (1-\pi)^t$. We use this latter function to show that any Geometric RV exhibits the *memory-less property*: $\Pr\{T > s + t \mid T > s\} = \Pr\{T > t\}$. The result is actually 'IFF' in the sense that in addition if a discrete RV has the memory-less property, then it must be a Geometric RV.**

**For T a Geometric RV, it is easy to show (pp. 244-5) that $E(T) = 1/\pi$ and $Var(T) = \frac{1-\pi}{\pi^2}$. The proofs use the infinite sum (and other derived infinite sums) we gave at the end of the Chapter 4 Notes.**

**Exponential RV's – In Chapter 7, we defined T as having the Exponential (θ) distribution (written $T \sim \mathcal{E}(\theta)$) provided its pdf is**

$$f(t) = \frac{1}{\theta} e^{-t/\theta} \quad \text{for } t > 0$$

**The unknown parameter is θ > 0. We also showed there that the cdf is $F(t) = \Pr\{T \le t\} = 1 - e^{-t/\theta}$, for t > 0. Hence, the survival function is $S(t) = \Pr\{T > t\} = e^{-t/\theta}$ for t > 0. It is simple then to show that the Exponential distribution also follows the memory-less property – it is the only continuous RV with this property (proof on p.248). We also showed previously that $E(T) = \theta$ and $var(T) = \theta^2$.**

**Poisson RV's – The discrete RV Y has the Poisson(λ) distribution if it has the pmf**

$$f_Y(y) = \frac{\lambda^y}{y!} e^{-\lambda}, \ y = 0, 1, 2, 3 \ldots,$$

**and we write Y~ P(λ). In order to prove that this pmf sums to one, we need to remind ourselves of the result from Calculus: $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. We also use this result to show that E(Y) = λ and $E(Y^2) = \lambda^2 + \lambda$; hence, var(Y) = λ. It's kind of interesting that this distribution has just one parameter (λ), and it is both the mean and the variance.**

**In the last chapter, we used the Normal distribution to approximate the Binomial distribution, but another approximation of the Binomial distribution can be obtained from the Poisson distribution – provided again that n is large and π is small. Equating means, we set λ = nπ. A general rule (p.255) says that the approximation is *excellent* for n ≥ 100 and λ = nπ ≤ 5, and is *very good* for n ≥ 100 and λ = nπ < 10.**

**Example 9.5 (p.256) – First Aid. There are n = 2000 spectators and the chance that any one individual needs first aid is π = 0.001 (one in a thousand). Let's use the Poisson distribution to approximate the probability that at *least 3* spectators need first aid treatment. Here, λ = nπ = 2, so the Poisson approximation should be excellent. We want Pr{K ≥ 3} = 1 – Pr{K ≤ 2} ≈ 1 – Pr{Y ≤ 2} (with Y ~ P(λ = 2)) = $1 - \{e^{-2} + 2e^{-2} + 2e^{-2}\} = 1 - 5e^{-2} = 0.3233$ or 32.33%. See p.256 to see how close this approximation is to the true value.**

**On p.257, two recursion relations are given as is a theorem justifying the Poisson approximation to the Binomial distribution. The proof of this result is based on the important limit from basic Calculus:**

$$\lim_{n\to\infty}\left(1+\frac{f(x)}{n}\right)^n = e^{f(x)}$$

**Also, the Exponential and Poisson distributions are connected via the following (p.260):**

**<u>Theorem 9.8</u> – Let T be an exponential random variable of waiting times for an event with mean waiting time of $\theta$ units. Let Y be the count of events that occur in a unit of time, $[0,1]$. Then, Y has a Poisson distribution with mean $\lambda = 1/\theta$.**

**A nice application is given in Example 9.6.**